AN ALTERNATIVE PROOF OF ELEZOVIĆ-GIORDANO-PEČARIĆ'S THEOREM

FENG QI AND BAI-NI GUO

Abstract. In the present note, an alternative proof is supplied for Theorem 1 in [N. Elezović, C. Giordano and J. Pečarić, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. **3** (2000), 239–252.].

1. Introduction

Let s and t be real numbers with $t-s\ne\pm1$. For $x\in(-\alpha,\infty)$, define

$$z_{s,t}(x)=\begin{cases}\left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}-x, & s\ne t,\\ e^{\psi(x+s)}-x, & s=t,\end{cases}\tag{1}$$

where $\alpha=\min\{s,t\}$,

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,\mathrm{d}\,t\tag{2}$$

for $x>0$ stands for the classical Euler's gamma function, and $\psi(x)$ denotes the psi or digamma function, the derivative of the logarithm $\ln\Gamma(x)$.

In order to bound the ratio of two gamma functions from both sides, N. Elezović, C. Giordano and J. Pečarić proved in [2, Theorem 1] the following monotonicity and convexity results of the function $z_{s,t}(x)$.

Theorem 1. *The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s|<1$ or concave and increasing for $|t-s|>1$.*

The explicit or implicit origins and background of this theorem may be traced back to [3, 5, 18, 20] and [6, Theorem 2]. This theorem or its special cases have been proved several times by different approaches in, for example, [1, 6, 8, 11, 16, 17, 18]. For detailed information on its history, please refer to the survey article [9] published as a preprint recently.

The purpose of this note is to supply an alternative proof for Theorem 1.

2. Lemmas

In order to prove Theorem 1 alternatively, the following lemmas are necessary.

2000 *Mathematics Subject Classification.* 26A48, 26A51, 26D20, 33B15.

Key words and phrases. an alternative proof, Elezović-Giordano-Pečarić's theorem, monotonicity, convexity, ratio of two gamma functions, convolution theorem of Laplace transforms.

The first author was partially supported by the China Scholarship Council.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

Lemma 1 ([7, p. 16]). *The polygamma functions $\psi^{(n)}(x)$ can be expressed for $x>0$ and $n\in\mathbb{N}$ as*

$$\psi^{(n)}(x)=(-1)^{n+1}\int_0^\infty\frac{t^n}{1-e^{-t}}e^{-xt}\,\mathrm{d}\,t. \tag{3}$$

Lemma 2 ([19]). *Let $f_i(t)$ for $i=1,2$ be piecewise continuous in arbitrary finite intervals included on $(0,\infty)$, suppose there exist some constants $M_i>0$ and $c_i\ge0$ such that $|f_i(t)|\le M_ie^{c_it}$ for $i=1,2$. Then*

$$\int_0^\infty\biggl[\int_0^t f_1(u)f_2(t-u)\,\mathrm{d}\,u\biggr]e^{-st}\,\mathrm{d}\,t=\int_0^\infty f_1(u)e^{-su}\,\mathrm{d}\,u\int_0^\infty f_2(v)e^{-sv}\,\mathrm{d}\,v. \tag{4}$$

Lemma 3. *For $u\in\mathbb{R}$ and $\beta>\alpha\ge0$ with $(\alpha,\beta)\ne(0,1)$, let*

$$q_{\alpha,\beta}(u)=\begin{cases}\dfrac{e^{-\alpha u}-e^{-\beta u}}{1-e^{-u}}, & u\ne0;\\ \beta-\alpha, & u=0.\end{cases} \tag{5}$$

(1) *The function $q_{\alpha,\beta}(u)$ is logarithmically convex for $\beta-\alpha>1$ and logarithmically concave for $0<\beta-\alpha<1$ on $(-\infty,\infty)$.*
(2) *For $\beta-\alpha>1$, the function*

$$Q_{s,t;\lambda}(u)=q_{\alpha,\beta}(u)q_{\alpha,\beta}(\lambda-u) \tag{6}$$

is increasing on $\bigl(\frac{\lambda}{2},\infty\bigr)$ and decreasing on $\bigl(-\infty,\frac{\lambda}{2}\bigr)$, where λ is any real constant; For $0<\beta-\alpha<1$, it is decreasing on $\bigl(\frac{\lambda}{2},\infty\bigr)$ and increasing on $\bigl(-\infty,\frac{\lambda}{2}\bigr)$.

Proof. It is clear that the function $q_{\alpha,\beta}(u)$ can be rewritten as

$$q_{\alpha,\beta}(u)=\frac{\sinh((\beta-\alpha)u/2)}{\sinh(u/2)}\exp\frac{(1-\alpha-\beta)u}{2}\triangleq p_{\alpha,\beta}\biggl(\frac{u}{2}\biggr).$$

Since the functions $q_{\alpha,\beta}(u)$ and $p_{\alpha,\beta}(u)$ are positive for $\beta>\alpha$, taking the logarithm of $p_{\alpha,\beta}(u)$ and differentiating yield

$$\begin{aligned}\ln p_{\alpha,\beta}(u)&=\ln\sinh((\beta-\alpha)u)-\ln\sinh u+(1-\alpha-\beta)u,\\ [\ln p_{\alpha,\beta}(u)]'&=(\beta-\alpha)\coth((\beta-\alpha)u)-\coth u-\alpha-\beta+1,\\ [\ln p_{\alpha,\beta}(u)]''&=\frac{1}{u^2}\biggl\{\biggl(\frac{u}{\sinh u}\biggr)^2-\biggl[\frac{(\beta-\alpha)u}{\sinh((\beta-\alpha)u)}\biggr]^2\biggr\}\\ &\triangleq\frac{[h(u)]^2-[h((\beta-\alpha)u)]^2}{u^2}.\end{aligned}$$

It is clear that the functions $h(u)$ and $[\ln p_{\alpha,\beta}(u)]''$ are even and the former is positive on $(-\infty,\infty)$, increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$. As a result,

(1) for $\beta-\alpha>1$, if $u>0$, then $(\beta-\alpha)u>u>0$ and $h((\beta-\alpha)u)<h(u)$, and so $[\ln p_{\alpha,\beta}(u)]''>0$ on $(0,\infty)$;
(2) for $\beta-\alpha>1$, if $u<0$, then $(\beta-\alpha)u<u<0$ and $h((\beta-\alpha)u)<h(u)$, and so $[\ln p_{\alpha,\beta}(u)]''>0$ on $(-\infty,0)$;
(3) for $0<\beta-\alpha<1$, if $u>0$, then $0<(\beta-\alpha)u<u$ and $h((\beta-\alpha)u)>h(u)$, and so $[\ln p_{\alpha,\beta}(u)]''<0$ on $(0,\infty)$;
(4) for $0<\beta-\alpha<1$, if $u<0$, then $0>(\beta-\alpha)u>u$ and $h((\beta-\alpha)u)>h(u)$, and so $[\ln p_{\alpha,\beta}(u)]''<0$ on $(-\infty,0)$.

From the obvious relationship $p_{\alpha,\beta}(u)=q_{\alpha,\beta}(2u)$ on $(-\infty,\infty)$, the logarithmically convex properties in Lemma 3 follows readily.

Taking the logarithm of $Q_{s,t;\lambda}(u)$ and differentiating give

$$[\ln Q_{s,t;\lambda}(u)]'=\frac{q'_{\alpha,\beta}(u)}{q_{\alpha,\beta}(u)}-\frac{q'_{\alpha,\beta}(\lambda-u)}{q_{\alpha,\beta}(\lambda-u)}.$$

For $\beta-\alpha>1$, by the logarithmic convexities of $q_{\alpha,\beta}(u)$, it follows that the function $\frac{q'_{\alpha,\beta}(u)}{q_{\alpha,\beta}(u)}$ is increasing and $\frac{q'_{\alpha,\beta}(\lambda-u)}{q_{\alpha,\beta}(\lambda-u)}$ is decreasing on $(-\infty,\infty)$; From the obvious fact that $[\ln Q_{s,t;\lambda}(u)]'|_{u=\lambda/2}=0$, it follows that $[\ln Q_{s,t;\lambda}(u)]'>0$ for $u>\frac{\lambda}{2}$ and $[\ln Q_{s,t;\lambda}(u)]'<0$ for $u<\frac{\lambda}{2}$; Hence, the function $Q_{s,t;\lambda}(u)$ is increasing for $u>\frac{\lambda}{2}$ and decreasing for $u<\frac{\lambda}{2}$. Similarly, for $0<\beta-\alpha<1$, the function $Q_{s,t;\lambda}(u)$ is decreasing for $u>\frac{\lambda}{2}$ and increasing for $u<\frac{\lambda}{2}$. The proof of Lemma 3 is proved. □

Lemma 4. *For $x\in(0,\infty)$,*

$$\ln x-\frac{1}{x}<\psi(x)<\ln x-\frac{1}{2x} \tag{7}$$

and

$$\frac{1}{x}+\frac{1}{2x^2}<\psi'(x)<\frac{1}{x}+\frac{1}{x^2}. \tag{8}$$

Proof. This may be derived easily from the fact [15, p. 82] that a completely monotonic function which is non-identically zero cannot vanish at any point on $(0,\infty)$ and the complete monotonicity obtained in [12, Theorem 2] and [13, Theorem 2]: The function $\psi(x)-\ln x+\frac{\alpha}{x}$ is completely monotonic on $(0,\infty)$ if and only if $\alpha\ge1$ and so is the function $\ln x-\frac{\alpha}{x}-\psi(x)$ if and only if $\alpha\le\frac12$. □

3. An alternative proof of Theorem 1

Since $z_{s,t}(x)=z_{t,s}(x)$, without loss of generality, we can assume $t>s\ge0$ and $t-s\ne1$ in what follows.

Differentiation of $z_{s,t}(x)$, utilization of (3) and application of Lemma 2 yield

$$z'_{s,t}(x)=\frac{[z_{s,t}(x)+x][\psi(x+t)-\psi(x+s)]}{t-s}-1, \tag{9}$$

$$\begin{aligned}
\frac{z''_{s,t}(x)}{z_{s,t}(x)+x}&=\left[\frac{\psi(x+t)-\psi(x+s)}{t-s}\right]^2+\frac{\psi'(x+t)-\psi'(x+s)}{t-s}\\
&=\left[\frac{1}{t-s}\int_s^t\psi'(x+u)\,\mathrm{d}\,u\right]^2+\frac{1}{t-s}\int_s^t\psi''(x+u)\,\mathrm{d}\,u\\
&=\left[\frac{1}{t-s}\int_s^t\int_0^\infty\frac{ve^{-(x+u)v}}{1-e^{-v}}\,\mathrm{d}\,v\,\mathrm{d}\,u\right]^2-\frac{1}{t-s}\int_s^t\int_0^\infty\frac{v^2e^{-(x+u)v}}{1-e^{-v}}\,\mathrm{d}\,v\,\mathrm{d}\,u\\
&=\left(\int_0^\infty\frac{ve^{-xv}}{1-e^{-v}}\cdot\frac{1}{t-s}\int_s^te^{-uv}\,\mathrm{d}\,u\,\mathrm{d}\,v\right)^2-\int_0^\infty\frac{v^2e^{-xv}}{1-e^{-v}}\cdot\frac{1}{t-s}\int_s^te^{-uv}\,\mathrm{d}\,u\,\mathrm{d}\,v\\
&=\left(\int_0^\infty\frac{e^{-xv}}{1-e^{-v}}\cdot\frac{e^{-sv}-e^{-tv}}{t-s}\,\mathrm{d}\,v\right)^2-\int_0^\infty\frac{ve^{-xv}}{1-e^{-v}}\cdot\frac{e^{-sv}-e^{-tv}}{t-s}\,\mathrm{d}\,v\\
&=\int_0^\infty\left[\frac{1}{(t-s)u}\int_0^uq_{s,t}(r)q_{s,t}(u-r)\,\mathrm{d}\,r-q_{s,t}(u)\right]ue^{-xu}\,\mathrm{d}\,u
\end{aligned}$$

$$= \int_0^\infty \left[\frac{1}{(t-s)u} \int_0^u Q_{s,t;u}(r) \, \mathrm{d}\, r - q_{s,t}(u) \right] u e^{-xu} \, \mathrm{d}\, u. \tag{10}$$

If $t - s > 1$, by the monotonicity of $Q_{s,t;\lambda}(u)$ in Lemma 3, it follows easily that

$$Q_{s,t;u}(r) \le Q_{s,t;u}(0) = Q_{s,t;u}(u) = q_{s,t}(0) q_{s,t}(u) = (t-s) q_{s,t}(u),$$

consequently, the bracketed term in the line (10) is negative on $(0, \infty)$, and so $z''_{s,t}(x) < 0$. If $0 < t - s < 1$, the similar argument leads to $z''_{s,t}(x) > 0$. The convex and concave properties of $z_{s,t}(x)$ are proved.

By the mean value theorem, it is immediate that

$$\begin{aligned} z'_{s,t}(x) + 1 &= \left[\left(\frac{\Gamma(x+t)}{\Gamma(x+s)} \right)^{1/(t-s)} \frac{\psi(x+t) - \psi(x+s)}{t-s} \right] \\ &= \frac{\psi(x+t) - \psi(x+s)}{t-s} \exp \frac{\ln \Gamma(x+t) - \ln \Gamma(x+s)}{t-s} \\ &= \psi'(x + \xi_1) e^{\psi(x+\xi_2)}, \quad \xi_i \in (s,t) \text{ for } i = 1, 2. \end{aligned}$$

By inequalities in (7) and (8), it is ready to obtain

$$\left[\frac{x+\xi_2}{x+\xi_1} + \frac{x+\xi_2}{2(x+\xi_1)^2} \right] \frac{1}{e^{1/(x+\xi_2)}} < z'_{s,t}(x) + 1 < \left[\frac{x+\xi_2}{x+\xi_1} + \frac{x+\xi_2}{(x+\xi_1)^2} \right] \frac{1}{e^{1/2(x+\xi_2)}}$$

which means $\lim_{x\to\infty} z'_{s,t}(x) = 0$. For $t - s > 1$, the conclusion that $z''_{s,t}(x) \le 0$ obtained above implies $z'_{s,t}(x)$ is decreasing, and so $z'_{s,t}(x) > 0$ and $z_{s,t}(x)$ is increasing. For $0 < t - s < 1$, the result that $z''_{s,t}(x) \ge 0$ obtained above implies $z'_{s,t}(x)$ is increasing, and so $z'_{s,t}(x) < 0$ and $z_{s,t}(x)$ is decreasing. The proof of Theorem 1 is complete.

4. Some remarks

Remark 1. The logarithmically convex properties of $q_{\alpha,\beta}(u)$ on $(-\infty, 0)$ in Lemma 3 of this paper corrects some mistakes appeared in [16, Lemma 1] and [17, Lemma 1]. However, these mistakes did not affect the correctness of the proof provided in [16, 17] for Theorem 1, since properties of $q_{\alpha,\beta}(u)$ on $(-\infty, 0)$ are idle there.

Remark 2. The logarithmically convex properties in Lemma 3 of this paper were also proved in [10] by using different techniques. Also see [4] and related references therein.

Remark 3. It is well-known that a positive and k-times differentiable function $f(x)$ is said to be k-log-convex (or k-log-concave, respectively) on an interval I with $k \ge 2$ if and only if $[\ln f(x)]^{(k)}$ exists and $[\ln f(x)]^{(k)} \ge 0$ (or $[\ln f(x)]^{(k)} \le 0$, respectively) on I. The 3-log-convex properties of $q_{\alpha,\beta}(u)$ were already obtained in [14, Theorem 1.1]: For $1 > \beta - \alpha > 0$, the function $q_{\alpha,\beta}(u)$ is 3-log-convex on $(0, \infty)$ and 3-log-concave on $(-\infty, 0)$; For $\beta - \alpha > 1$, it is 3-log-concave on $(0, \infty)$ and 3-log-convex on $(-\infty, 0)$.

References

[1] Ch.-P. Chen, *Monotonicity and convexity for the gamma function*, J. Inequal. Pure Appl. Math. **6** (2005), no. 4, Art. 100; Available online at `http://jipam.vu.edu.au/article.php?sid=574`.

[2] N. Elezović, C. Giordano and J. Pečarić, *The best bounds in Gautschi's inequality*, Math. Inequal. Appl. **3** (2000), 239–252.

[3] W. Gautschi, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. Phys. **38** (1959/60), 77–81.

[4] B.-N. Guo and F. Qi, *Properties and applications of a function involving exponential functions*, Commun. Pure Appl. Anal. **8** (2009), no. 4, in press.

[5] D. Kershaw, *Some extensions of W. Gautschi's inequalities for the gamma function*, Math. Comp. **41** (1983), 607–611.

[6] I. Lazarević and A. Lupaş, *Functional equations for Wallis and Gamma functions*, Publ. Elektrotehn. Fak. Univ. Beograd. Ser. Electron. Telecommun. Automat. No. **461-497** (1974), 245–251.

[7] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer, Berlin, 1966.

[8] F. Qi, *A completely monotonic function involving divided differences of psi and polygamma functions and an application*, RGMIA Res. Rep. Coll. **9** (2006), no. 4, Art. 8; Available online at `http://www.staff.vu.edu.au/rgmia/v9n4.asp`.

[9] F. Qi, *Bounds for the ratio of two gamma functions*, RGMIA Res. Rep. Coll. **11** (2008), no. 3, Art. 1; Available online at `http://www.staff.vu.edu.au/rgmia/v11n3.asp`.

[10] F. Qi, *Monotonicity and logarithmic convexity for a class of elementary functions involving the exponential function*, RGMIA Res. Rep. Coll. **9** (2006), no. 3, Art. 3; Available online at `http://www.staff.vu.edu.au/rgmia/v9n3.asp`.

[11] F. Qi, *The best bounds in Kershaw's inequality and two completely monotonic functions*, RGMIA Res. Rep. Coll. **9** (2006), no. 4, Art. 2; Available online at `http://www.staff.vu.edu.au/rgmia/v9n4.asp`.

[12] F. Qi, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, Integral Transforms Spec. Funct. **18** (2007), no. 7, 503–509.

[13] F. Qi, *Three classes of logarithmically completely monotonic functions involving gamma and psi functions*, RGMIA Res. Rep. Coll. **9** (2006), Suppl., Art. 6; Available online at `http://www.staff.vu.edu.au/rgmia/v9(E).asp`.

[14] F. Qi, *Three-log-convexity for a class of elementary functions involving exponential function*, J. Math. Anal. Approx. Theory **1** (2006), no. 2, 100–103.

[15] F. Qi, B.-N. Guo and Ch.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Aust. Math. Soc. **80** (2006), 81–88.

[16] F. Qi, B.-N. Guo and Ch.-P. Chen, *The best bounds in Gautschi-Kershaw inequalities*, Math. Inequal. Appl. **9** (2006), no. 3, 427–436.

[17] F. Qi, B.-N. Guo and Ch.-P. Chen, *The best bounds in Gautschi-Kershaw inequalities*, RGMIA Res. Rep. Coll. **8** (2005), no. 2, Art. 17; Available online at `http://www.staff.vu.edu.au/rgmia/v8n2.asp`.

[18] G. N. Watson, *A note on gamma functions*, Proc. Edinburgh Math. Soc. **11** (1958/1959), no. 2, Edinburgh Math Notes No. 42 (misprinted 41) (1959), 7–9.

[19] E. W. Weisstein, *Laplace Transform*, From MathWorld—A Wolfram Web Resource; Available online at `http://mathworld.wolfram.com/LaplaceTransform.html`.

[20] J. G. Wendel, *Note on the gamma function*, Amer. Math. Monthly **55** (1948), no. 9, 563–564.

(F. Qi) Research Institute of Mathematical Inequality Theory, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com`

URL: `http://qifeng618.spaces.live.com`

(B.-N. Guo) School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo City, Henan Province, 454010, China

E-mail address: `bai.ni.guo@gmail.com, bai.ni.guo@hotmail.com`

URL: `http://guobaini.spaces.live.com`